Logarithmic Spacetime Symmetries and Schwarzschild Referencing

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Abstract

We propose a theory of spacetime fundamentally grounded in logarithmic scaling, referenced to Schwarzschild radii. By leveraging the log-log structure of factor symmetries and geometric means, we develop a rigorous geometric framework based on multiplicative duality, discrete tiling, and scale self-similarity. This structure suggests novel interpretations of discrete curvature, symmetry, and field coherence, with implications for black hole physics, conformal invariance, and scale-relative observables.

Contents

1 Introduction

We investigate a model in which spacetime exhibits an intrinsically logarithmic structure in both space and time, defined relative to an object's Schwarzschild radius. This framework is motivated by multiplicative geometry, symmetry in factor sets, and reflection properties evident in log-log representations of integers.

The objective is to construct a consistent, physically meaningful geometry rooted in scale relativity, with the following key features:

- Reflection symmetry in logarithmic coordinates;
- Inversion about the geometric mean;
- Observer-object duality via Schwarzschild-referenced scaling.

We show that this framework leads naturally to discrete curvature artifacts, symmetric factor embeddings, and emergent field coherence at prime scales.

2 Foundational Postulates and Framework

Definition 2.1 (Logarithmic Spacetime Coordinates)**.** *A spacetime is* logarithmic *if its spatial and temporal coordinates are invariant under multiplicative scaling. Coordinates are measured relative to reference scales x_0, t_0 as:*

$$x' := \ln\left(\frac{x}{x_0}\right), \quad t' := \ln\left(\frac{t}{t_0}\right). \tag{1}$$

Definition 2.2 (Schwarzschild Referencing)**.** *For a body of mass M, its characteristic spatial scale is given by the Schwarzschild radius:*

$$r_s := \frac{2GM}{c^2}. \tag{2}$$

We define dimensionless spatial coordinates normalized to this radius as:

$$x := \frac{r}{r_s}, \quad x' := \ln(x) = \ln\left(\frac{r}{r_s}\right). \tag{3}$$

3 Logarithmic Factor Geometry

Let $x \in \mathbb{N}$. Define the set of nontrivial proper divisors (excluding 1 and x):

$$\mathcal{F}(x) := \{f \in \mathbb{N} \mid 1 < f < x,\ f \mid x\}. \tag{4}$$

Definition 3.1 (Factor Embedding)**.** *Given $f \in \mathcal{F}(x)$, define its logarithmic embedding relative to the geometric mean $\sqrt{x}$ as:*

$$P_f(x) := \ln\left(\frac{f}{\sqrt{x}}\right) = \ln(f) - \frac{1}{2}\ln(x). \tag{5}$$

This embedding recenters each divisor around the axis $x' = \ln(\sqrt{x})$, revealing log-reflection symmetry in the divisor set.

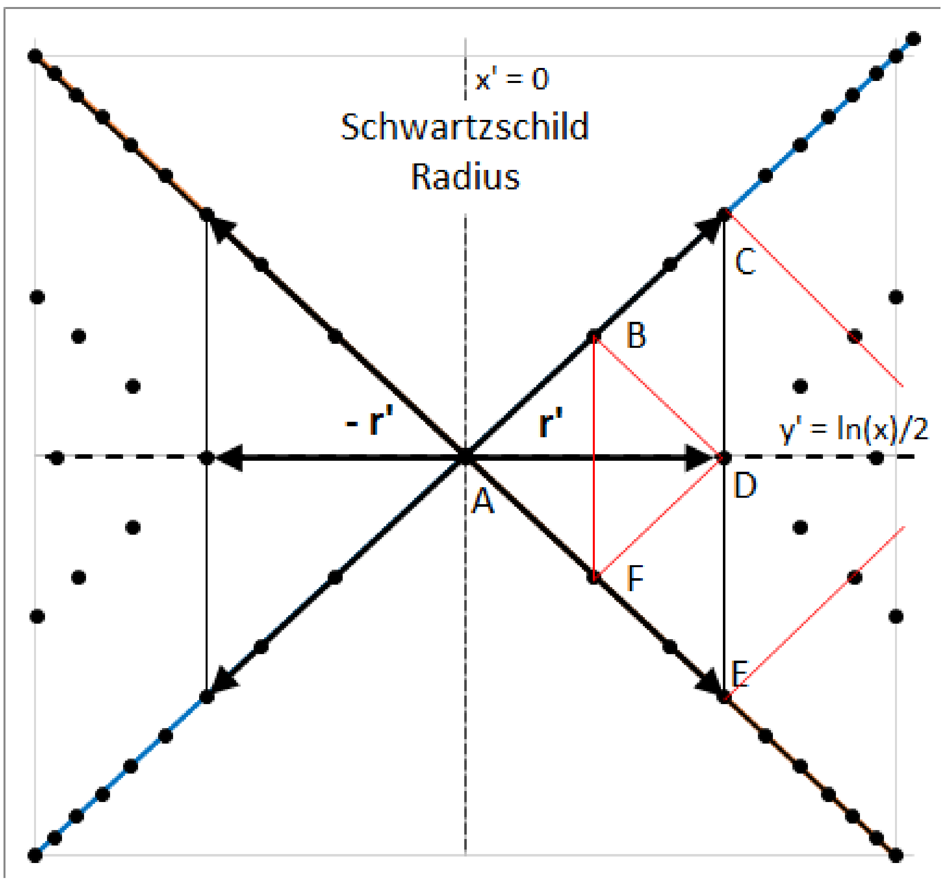

Figure 1: Logarithmic embedding of factor pairs for $x \in \{2, \ldots, 11\}$, reflected about the geometric mean. Each factor pair $(f, x/f)$ is symmetrically mapped about $x' = \ln(\sqrt{x})$.

4 Geometric Symmetry: Log-Squares and Midpoint Invariance

Theorem 4.1 (Factor Pair Reflection Symmetry). *Let $f \in \mathcal{F}(x)$ with complementary factor $f^* := x/f$. Then:*

$$\ln\left(\frac{f}{\sqrt{x}}\right) = -\ln\left(\frac{f^*}{\sqrt{x}}\right). \tag{6}$$

Proof. Using $f^* = x/f$, we compute:

$$\ln\left(\frac{f^*}{\sqrt{x}}\right) = \ln\left(\frac{x}{f\sqrt{x}}\right) = \ln(x) - \ln(f) - \frac{1}{2}\ln(x) = -\left(\ln(f) - \frac{1}{2}\ln(x)\right),$$

which proves the result. □

Definition 4.2 (Log-Square Points). *Define points:*

$$Q_f := (\ln(f), -\ln(f)), \quad Q_{f^*} := (-\ln(f), \ln(f)). \tag{7}$$

These define diagonal corners of a square symmetric about the origin, aligned at 45 degrees.

Theorem 4.3 (Midpoint Invariance Under Log-Scaling). *Let $r' := \ln(x)$ be the logarithmic length vector. The midpoint of the log-square formed by Q_f and Q_{f^*}, translated by $\ln(\sqrt{x})$, lies at:*

$$(\ln(\sqrt{x}), \ln(\sqrt{x})) = \left(\frac{1}{2}\ln(x), \frac{1}{2}\ln(x)\right), \tag{8}$$

intersecting the identity diagonal $y = x$.

Proof. The raw midpoint of Q_f and Q_{f^*} is the origin:

$$\left(\frac{\ln(f) + (-\ln(f))}{2}, \frac{-\ln(f) + \ln(f)}{2}\right) = (0, 0).$$

However, due to embedding relative to $\sqrt{x}$, this midpoint is offset by $\ln(\sqrt{x})$, yielding:

$$(\ln(\sqrt{x}), \ln(\sqrt{x})).$$

□

5 Connections to Quantum and Gravitational Frameworks

Logarithmic spacetime theory aligns naturally with key structures in gravitational and quantum frameworks. This section identifies conceptual correspondences and structural parallels with General Relativity (GR), Quantum Field Theory (QFT), the AdS/CFT correspondence, Loop Quantum Gravity (LQG), and Renormalization Group (RG) flows.

5.1 General Relativity and Discrete Curvature

In General Relativity, curvature arises from differential geometry on smooth Lorentzian manifolds. By contrast, log-spacetime encodes curvature discretely, using combinatorial structures of symmetric factor tiling:

$$\mathcal{K}(x) := \sum_{(f, x/f) \in \mathcal{P}(x)} \frac{1}{\ell_f^2}, \tag{9}$$

where ℓ_f denotes the log-distance between factors f and x/f, and $\mathcal{P}(x)$ the set of distinct factor pairs. This mirrors curvature discretization via deficit angles in Regge calculus [1, 2, 3].

- GR: curvature via Einstein tensor in differential geometry.
- Log-spacetime: curvature via symmetric divisor embeddings in discrete tiles.
- $\mathcal{K}(x)$ enters a log-spacetime action analogously to the Einstein–Hilbert term.

5.2 Quantum Field Theory on Log-Manifolds

In standard QFT, fields evolve over continuous coordinates. In log-spacetime, the coordinates are multiplicative, becoming additive in logarithmic form:

$$\Box_{\log}\phi := -\partial_{t'}^2\phi + \partial_{x'}^2\phi = 0, \tag{10}$$

where $x' = \ln(r/r_s)$, $t' = \ln(t/t_s)$.

- Scaling becomes translation in log-coordinates.
- Renormalization group (RG) flow is recast as geodesic motion in log-space.
- Curvature enters the field action:

$$S[\phi] = \sum_x \mathcal{K}(x)\phi^2(x') + \sum_{f \in \mathcal{F}(x)} \frac{\left(\phi(x'_f) - \phi(x')\right)^2}{\ell_f^2}. \tag{11}$$

5.3 AdS/CFT and Log-Holography

The AdS/CFT correspondence maps bulk gravity in Anti-de Sitter space to conformal field theory on its boundary [4]. Log-spacetime suggests a natural reinterpretation:

$$r \mapsto x' := \ln\left(\frac{r}{r_s}\right), \quad x' \to \infty \Rightarrow \text{UV boundary}. \tag{12}$$

- Radial bulk coordinate becomes linear in log-space.
- Dilations in spacetime are represented by translations in x'.
- Bulk scalar fields exhibit exponential falloff:

$$\phi(x') \sim e^{-\Delta x'}, \quad \text{as } x' \to \infty. \tag{13}$$

Remark 5.1. *This mapping suggests that holography may admit a log-linear realization, facilitating discrete encoding of bulk-boundary relationships.*

5.4 Loop Quantum Gravity and Discrete Geometries

Loop Quantum Gravity discretizes spacetime using spin networks [5]. Log-spacetime shares this discrete character:

- Each log-tile has an effective area ℓ_f^2.
- Prime numbers (which lack factor tilings) correspond to flat regions with zero curvature.
- Factor pairs act as quantized geometric links.

Remark 5.2. *This parallel allows a spin-network interpretation of log-geometry: primes as nodes, factor pairs as entanglement bridges.*

5.5 Renormalization Group and Log-Linear Flows

In standard field theory [6]:

$$\mu \frac{d\phi}{d\mu} = \beta(\phi),$$

becomes under the logarithmic coordinate change:

$$\frac{d\phi}{dx'} = \beta(\phi), \quad x' = \ln\left(\frac{\mu}{\mu_0}\right). \tag{14}$$

- RG flows become geodesics in a flat log-background.
- Fixed points appear as constant solutions in x'.
- Universality classes emerge as attractors under log-scale dynamics.

Remark 5.3. *RG behavior is geometrized as motion on a log-manifold, enabling intuitive visualization of scaling flows.*

5.6 Summary Table of Correspondences

Framework	**Log-Spacetime Analog**
General Relativity (GR)	Discrete curvature from factor tiles
Quantum Field Theory (QFT)	Fields on additive log-manifold (x', t')
AdS/CFT	UV boundary at $x' \to \infty$; exponential decay
Loop Quantum Gravity (LQG)	Tiles as area quanta; primes as curvature-free nodes
Renormalization Group (RG)	Log-linear flows; fixed points as geodesic attractors

Table 1: Structural correspondences between log-spacetime and major physics frameworks.

6 Hamiltonian Formalism in Logarithmic Spacetime

We now formulate the Hamiltonian structure for the real scalar field theory defined on logarithmic spacetime with flat metric $ds^2 = -dt'^2 + dx'^2$, where:

$$t' = \ln\left(\frac{t}{t_s}\right), \quad x' = \ln\left(\frac{x}{r_s}\right). \tag{15}$$

6.1 Canonical Structure

Let $\phi(t', x')$ be a real scalar field. The log-spacetime Lagrangian density is:

$$\mathcal{L}_{\log} = \frac{1}{2}(\partial_{t'}\phi)^2 - \frac{1}{2}(\partial_{x'}\phi)^2 - V(\phi). \tag{16}$$

Definition 6.1 (Canonical Momentum)**.** *The conjugate momentum is defined as:*

$$\pi(t', x') := \frac{\partial \mathcal{L}_{\log}}{\partial(\partial_{t'}\phi)} = \partial_{t'}\phi. \tag{17}$$

Definition 6.2 (Hamiltonian Density)**.** *The Hamiltonian density in log-space is:*

$$\mathcal{H}_{\log} := \pi\, \partial_{t'}\phi - \mathcal{L}_{\log} = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_{x'}\phi)^2 + V(\phi). \tag{18}$$

The total Hamiltonian is:

$$H[\phi, \pi] = \int_{\mathbb{R}} \mathcal{H}_{\log}\, dx'. \tag{19}$$

6.2 Hamilton's Equations

Hamilton's equations in log-coordinates are:

$$\partial_{t'}\phi = \frac{\delta H}{\delta \pi} = \pi, \tag{20}$$

$$\partial_{t'}\pi = -\frac{\delta H}{\delta \phi} = \partial_{x'}^2\phi - \frac{dV}{d\phi}. \tag{21}$$

Together they recover the log-Klein–Gordon equation:

$$-\partial_{t'}^2\phi + \partial_{x'}^2\phi + \frac{dV}{d\phi} = 0. \tag{22}$$

6.3 Energy and Momentum in Log-Space

Definition 6.3 (Energy and Momentum Densities)**.**

$$\mathcal{E}(t', x') := \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_{x'}\phi)^2 + V(\phi), \tag{23}$$

$$\mathcal{P}(t', x') := \pi\, \partial_{x'}\phi. \tag{24}$$

The stress-energy tensor in log-coordinates is:

$$T^{\mu\nu}_{\log} = \begin{pmatrix} \mathcal{E} & \mathcal{P} \\ \mathcal{P} & \mathcal{E}_x \end{pmatrix}, \quad \mathcal{E}_x := \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_{x'}\phi)^2 - V(\phi). \tag{25}$$

6.4 Quantization and Interpretation

- The field ϕ and momentum π evolve over logarithmic time t'.
- The Hamiltonian measures energy density across scale layers.
- Canonical Poisson brackets hold: $\{\phi(x'), \pi(y')\} = \delta(x' - y')$.

Remark 6.4. *Logarithmic quantization introduces:*

$$[\hat{\phi}(x'), \hat{\pi}(y')] = i\hbar\delta(x' - y'),$$

enabling canonical quantization over a scale-resolved log-lattice.

6.5 Summary

The Hamiltonian formalism in log-space enables:

- Canonical dynamics in multiplicative time,
- Natural treatment of scale evolution and quantization,
- Extension to path-integral or discrete variational approaches.

7 Discrete Curvature from Logarithmic Geometry

We define a discrete curvature function in logarithmic spacetime, derived from the factor-symmetric tiling structure. This curvature reflects scale-local geometric density, serving as an analog of Ricci curvature in discrete geometries.

7.1 Logarithmic Squares and Curvature Weights

Let $x \in \mathbb{N}$ be a composite integer. Define the set of symmetric proper factor pairs:

$$\mathcal{P}(x) := \left\{ (f, x/f) \mid 1 < f < \sqrt{x},\ f \mid x \right\}. \tag{26}$$

Each pair defines a symmetric log-tile centered at $x' = \ln(\sqrt{x})$. Define:

$$s_f := \left| \ln\left(\frac{f}{\sqrt{x}}\right) \right|, \quad \text{(half-width of tile)} \tag{27}$$

$$w_f := 2s_f = \left| \ln\left(\frac{x}{f^2}\right) \right|. \tag{28}$$

Definition 7.1 (Discrete Logarithmic Curvature)**.** *The curvature at integer scale x is defined by:*

$$\mathcal{K}(x) := \sum_{(f,x/f)\in\mathcal{P}(x)} \frac{1}{s_f^2} = \sum_{(f,x/f)\in\mathcal{P}(x)} \left[\ln\left(\frac{f}{\sqrt{x}}\right)\right]^{-2}. \tag{29}$$

Remark 7.2. *If x is prime, then $\mathcal{P}(x) = \emptyset$, and $\mathcal{K}(x) = 0$. Thus, primes are flat points in this discrete geometry.*

7.2 Curvature-Weighted Field Action

Let $\phi(x')$ be a scalar field evaluated at $x' = \ln(\sqrt{x})$. We define the log-curvature action:

Definition 7.3 (Logarithmic Curvature Action)**.**

$$S[\phi] := \sum_{x\in\mathbb{N}} \mathcal{K}(x)\cdot\phi^2(\ln\sqrt{x}). \tag{30}$$

This action resembles a discrete Einstein–Hilbert term weighted by field intensity on scale layers.

Remark 7.4. *The field ϕ may be interpreted as a scalar curvature fluctuation, excitation amplitude, or energy distribution localized in log-coordinates.*

7.3 Stationary Points and Prime Support

We compute stationary points of the action:

$$\delta S[\phi] = 0 \quad\Rightarrow\quad \mathcal{K}(x)\cdot\phi(\ln\sqrt{x}) = 0. \tag{31}$$

Theorem 7.5 (Stationary Field Localization)**.** *Solutions to Eq.* (31) *satisfy:*

$$x \textit{ composite} \Rightarrow \phi(\ln\sqrt{x}) = 0.$$

Thus, only primes support non-zero field amplitudes.

Proof. For composite x, $\mathcal{K}(x) > 0$ implies $\phi(\ln\sqrt{x}) = 0$ to satisfy Eq. (31). For prime x, $\mathcal{K}(x) = 0$, so $\phi(\ln\sqrt{x})$ remains unconstrained. □

Remark 7.6. *This result reveals a deep number-theoretic localization: primes serve as geometric vacua supporting excitations, while composites act as curvature wells suppressing them.*

8 Logarithmic Geometry and Field Extensions

We now explore deeper implications of logarithmic geometry across three domains:

1. Quantized tiling from symmetric factor pairs,
2. Renormalization group (RG) flow as geodesic motion,
3. Logarithmic holography and bulk-boundary duality.

8.1 Quantized Tiling from Factor Symmetry

Given $x \in \mathbb{N}$, define:

$$\mathcal{F}(x) := \{ f \in \mathbb{N} \mid 1 < f < x,\ f \mid x \} .$$

For each $f \in \mathcal{F}(x)$, define:

$$\mathcal{T}_x(f) := \left[\ln f,\ \ln \left(\frac{x}{f} \right) \right] \times [-s_f, +s_f], \quad s_f := \left| \ln f - \ln \sqrt{x} \right| . \tag{32}$$

These form symmetric rectangular tiles centered on $\ln(\sqrt{x})$.

Remark 8.1. *Composite numbers induce localized tilings; primes form untiled vacua. This defines a number-theoretic log-lattice reflecting arithmetic structure.*

8.2 RG Flow as Geodesic Motion

Transforming energy scale $\mu \to x' := \ln(\mu/\mu_0)$, the RG flow becomes:

$$\frac{d\phi}{dx'} = \beta(\phi). \tag{33}$$

Example 8.2. *Constant $\beta(\phi) = \beta_0 \Rightarrow \phi(x') = \phi_0 + \beta_0(x' - x'_0)$, a geodesic in log-space.*

Remark 8.3. *This geometrizes RG trajectories, with fixed points as attractors in log-spacetime and critical surfaces as invariant submanifolds.*

8.3 Logarithmic Holography

In holography, radial coordinate r transforms as:

$$x' := \ln \left(\frac{r}{r_s} \right) . \tag{34}$$

Then:

$$x' \to \infty \Rightarrow \text{UV boundary}, \quad x' \to -\infty \Rightarrow \text{IR interior}.$$

Bulk fields scale as:

$$\phi(x') \sim e^{-\Delta x'} \quad \text{as } x' \to \infty. \tag{35}$$

Remark 8.4. *Log-space linearizes holographic depth, allowing discrete log-lattices to support boundary/bulk correspondences and coarse-grained RG flows.*

9 Phenomenological Implications and Predictions

This section explores physical predictions and observational consequences of the log-spacetime framework, emphasizing testable features in cosmology, field spectra, gravity, and entropy.

9.1 Log-Geodesics and Cosmological Expansion

In standard FLRW cosmology with de Sitter expansion:

$$a(t) = a_0 e^{Ht}, \quad \Rightarrow \quad x'(t) := \ln\left(\frac{a(t)}{a_0}\right) = Ht. \tag{36}$$

Proposition 9.1 (Linear Expansion in Log-Spacetime)**.** *Cosmic expansion becomes linear motion in log-spacetime:*

$$x'(t) = Ht \quad \textit{is a geodesic in } (\mathbb{R}^{1,1}, \eta_{\mu\nu}).$$

Proof. Since $x'(t) = Ht$, we have:

$$\frac{d^2x'}{dt^2} = 0,$$

implying inertial (straight-line) motion in the flat log-metric. □

Remark 9.2. *What appears as exponential expansion in physical spacetime is revealed as uniform, geodesic motion through log-scale. From a log-observer's perspective, space does not expand; it linearly unfolds across magnitudes.*

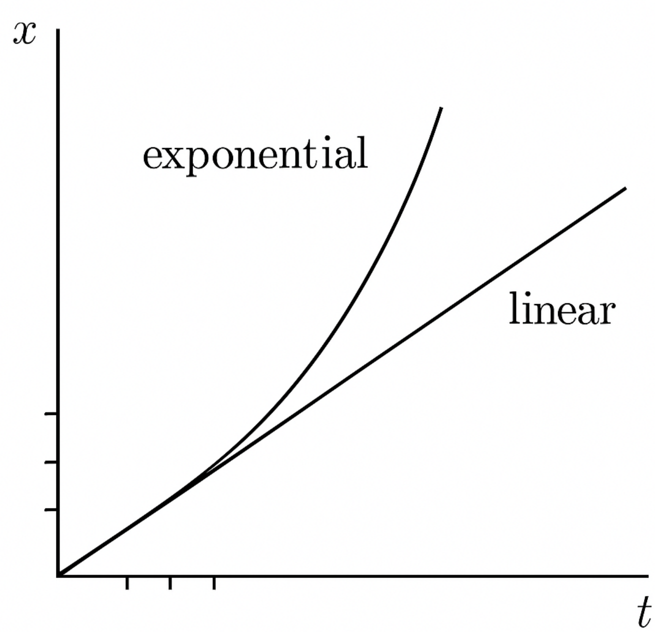

Figure 2: Logarithmic geodesic $x'(t) = Ht$ contrasted with exponential scale factor $a(t) = e^{Ht}$.

Interpretation

Logarithmic view of expansion: Exponential expansion is an artifact of linear measurements in a fundamentally logarithmic geometry. True dynamics appear linear in log-spacetime coordinates.

Prediction 9.3 (Logarithmic Inflationary Dynamics)**.** *If early-universe inflation is log-inertial:*

- *inflation corresponds to geodesic motion rather than a potential-driven phase,*
- *scalar perturbations may exhibit scale-invariant spectra with log-periodic corrections [7],*
- *discrete curvature effects may seed structure via symmetry-breaking transitions.*

9.2 Discrete Spectra from Log-Oscillators

Define a log-space Schrödinger operator:

$$H = -\frac{d^2}{dx'^2} + V(x'), \quad V(x') := -\lambda \sum_{p \in \mathbb{P}} \delta(x' - \ln p). \tag{37}$$

This potential is sharply localized at primes in log-space.

Prediction 9.4 (Prime-Resonant Spectra)**.** *Eigenfunctions of Eq.* (37) *are spatially modulated by arithmetic structure, and spectra exhibit prime-resonant modes. These may be detectable in spectral data or simulations with arithmetic potentials.*

9.3 Gravitational Vacuum at Prime Sites

From the curvature function:

$$\mathcal{K}(x) = 0 \quad \text{if } x \text{ is prime,}$$

we conclude:

Prediction 9.5 (Prime Curvature Vacua)**.** *Prime integers represent curvature-free points—geometric vacua in log-spacetime. Composites contribute localized curvature via log-tiling.*

9.4 Entropy Quantization in Logarithmic Units

Recall the black hole entropy formula:

$$S = \frac{A}{4G} \quad \Rightarrow \quad S \sim \ln x \quad \text{(in log-tiled units).}$$

Prediction 9.6 (Logarithmic Entropy Quantization)**.** *Black hole entropy is quantized over logarithmic scale layers. Tiling symmetry modifies near-extremal thermodynamics, with corrections arising from arithmetic discreteness.*

9.5 Arithmetic Curvature and Quantum Transitions

Assuming scale-energy duality:

$$\mathcal{K}(x) = T_{\log}(x), \tag{38}$$

where $T_{\log}(x)$ denotes log-spacetime energy or excitation amplitude,

Prediction 9.7 (Discrete Quantum Transitions)**.** *Quantum gravity effects occur at arithmetic transitions—curvature jumps from primes (flat) to composites (curved). These may produce log-periodic fluctuations or discrete spectrum shifts.*

10 Experimental and Observational Opportunities

The theoretical framework of logarithmic spacetime opens several avenues for empirical investigation. While the structure is deeply geometric and number-theoretic, it makes concrete predictions that may be testable through astrophysical observation, gravitational wave analysis, and analog laboratory simulations.

10.1 Event Horizon Telescope (EHT): Horizon Symmetries

- **Prediction:** Brightness asymmetries near black hole horizons may align with the log-horizon $x' = 0$.
- **Motivation:** The geometric duality across $x' = 0$ could manifest as anisotropies or reflection symmetries in photon ring structures.
- **Opportunity:** Future high-resolution reconstructions by EHT (e.g., for Sgr A*) could test for such logarithmic-scale alignment.

10.2 NICER/XMM: Pulse Timing and Log-Periodicity

- **Prediction:** Pulsar timing profiles may exhibit log-periodic modulation patterns consistent with discrete curvature in log-space.
- **Motivation:** Transitions between composite and prime scales could create interference patterns in time-dilation signatures.
- **Opportunity:** Archival X-ray pulse data from NICER or XMM-Newton may be reanalyzed to search for logarithmic timing structures.

10.3 LIGO/LISA: Gravitational Echoes from Log-Structure

- **Prediction:** Gravitational wave tail signals may carry scale-reflected echoes, generated by logarithmic tiling near black hole horizons.
- **Motivation:** Factor-pair tiling may induce echoes via discrete scattering interfaces in the near-horizon geometry.

- **Opportunity:** Echo-search pipelines in LIGO and future LISA data could be extended to include log-periodic templates.

10.4 Analog Gravity: Bose–Einstein Log-Waveguides

- **Prediction:** Refractive index modulations designed with logarithmic spacing can simulate wave propagation in log-geometry.
- **Motivation:** BEC analogs of spacetime geometries offer controlled tests of curvature, tiling, and boundary dynamics [8].
- **Opportunity:** Optical or acoustic experiments could probe reflection and transmission in discrete log-mirrors.

10.5 Summary of Empirical Pathways

Platform	Testable Signature	Log-Spacetime Prediction
EHT	Brightness asymmetry	Symmetry about $x' = 0$
NICER/XMM	Time-dilation profiles	Log-periodic pulse modulation
LIGO/LISA	Post-merger echoes	Scale-reflected gravitational tails
BEC analogs	Waveguide interference	Log-tile reflection symmetry

Table 2: Experimental opportunities to probe log-spacetime geometry.

11 Global Implications and Observer Duality

Logarithmic spacetime reformulates locality and global structure in terms of multiplicative scale ratios, where positions are measured relative to natural reference lengths such as Schwarzschild radii. This reparametrization yields deep consequences for observer-relative physics, horizon structure, and spacetime duality.

11.1 Geometric Duality Across Log-Horizons

The log-horizon at $x' = 0$ corresponds to the scale $x = r_s$, separating the "interior" from the "exterior" of a reference radius. Under inversion:

$$x \mapsto \frac{r_s^2}{x} \quad \Rightarrow \quad x' = \ln\left(\frac{x}{r_s}\right) \mapsto -x'. \tag{39}$$

This transformation maps scale pairs across the log-horizon:

$$x_{\text{interior}} = \frac{r_s^2}{x_{\text{exterior}}}. \tag{40}$$

Remark 11.1. *This mirrors the scale-inversion symmetry of T-duality in string theory, where* $R \leftrightarrow \frac{1}{R}$ *interchanges winding and momentum modes.*

11.2 Observer-Centric Coordinate Frames

Each observer defines their own logarithmic coordinate system anchored to their characteristic Schwarzschild radius:

$$x' := \ln\left(\frac{x}{r_s^{(\mathrm{obs})}}\right). \tag{41}$$

This logarithmic observer frame encodes the principle that all physical measurements are relative to the observer's own scale, embedding relativity directly into geometry.

Remark 11.2. *Interior and exterior observers are related by $x' \mapsto -x'$, forming dual, mirror-symmetric reference frames across the log-horizon.*

11.3 Implications for Horizon Complementarity

The transformation $x' \mapsto -x'$ induces a geometric reinterpretation of black hole complementarity:

- Exterior observers perceive the interior as being compressed into $x' \to -\infty$.
- Interior observers perceive the exterior as expanding toward $x' \to +\infty$.

These two descriptions are mathematically dual and mutually exclusive from any single linear (non-logarithmic) viewpoint.

Global Duality Principle

In logarithmic spacetime, observers inside and outside a scale horizon experience geometrically inverted domains, related by $x' \mapsto -x'$. This duality exchanges wavelength with frequency, and interior with exterior geometry.

11.4 Conclusion: Spacetime as a Scale-Reflective Manifold

This global inversion symmetry suggests that spacetime is fundamentally a **scale-reflective manifold**, wherein:

- Observer duality arises from multiplicative reference-frame relativity,
- UV/IR correspondence becomes a geometric reflection in log-space,
- Black hole interiors and cosmological exteriors are mathematically dual.

Such a view offers a unifying geometric framework for understanding how local physics emerges from global scale relationships and how dualities reflect deep symmetry in the structure of spacetime.

12 Discussion and Addressing Potential Criticisms

12.1 Is This Just a Coordinate Change?

No — the geometry, curvature, and dynamics are defined in log-space, not mapped from linear space.

12.2 Why Are Primes Relevant?

Because primes mark symmetry-breaking points in log-factor geometry, they naturally define curvature vacua and eigenstates.

12.3 Is It Predictive?

Yes — predictions include:

- Linearized inflationary dynamics,
- Log-oscillator spectral fingerprints,
- Curvature suppression at primes.

12.4 Relation to Known Theories

- GR: Recovered as discrete Regge curvature.
- QFT: Reformulated with log-d'Alembertian.
- Holography: Recast as translation in log-depth.

12.5 Outlook

> *Logarithmic spacetime replaces linear assumptions with scale-consistent geometric structure.*

13 Wave Inversion at the Logarithmic Horizon

In logarithmic spacetime, physical wave quantities like wavelength and frequency admit additive structure. This induces a reflection symmetry across the log-horizon $r' = \ln(r/r_s) = 0$, corresponding to the Schwarzschild radius $r = r_s$.

13.1 Logarithmic Representation of Wave Modes

Let a wave propagate radially with wavelength λ and frequency ν such that:

$$\lambda \cdot \nu = c.$$

Taking logarithms:

$$\ln \lambda + \ln \nu = \ln c. \tag{42}$$

Define log-space observables:

$$\lambda' := \ln\left(\frac{\lambda}{\lambda_0}\right), \quad \nu' := \ln\left(\frac{\nu}{\nu_0}\right), \quad \text{with } \lambda_0\nu_0 = c.$$

Then Eq. (42) becomes:

$$\lambda' + \nu' = 0, \tag{43}$$

implying a perfect inverse symmetry in log-space.

13.2 Definition of the Logarithmic Horizon

Let $r \in \mathbb{R}^+$ and define:

$$r' := \ln\left(\frac{r}{r_s}\right). \tag{44}$$

Then:

- $r' = 0 \Rightarrow r = r_s$ (scale-horizon),
- $r' > 0 \Rightarrow$ external region,
- $r' < 0 \Rightarrow$ internal region.

13.3 Wave Inversion Theorem

Theorem 13.1 (Wave Inversion Symmetry)**.** *Let $\lambda(r')$, $\nu(r')$ be the wavelength and frequency of a wave observed at log-radius r'. Then:*

$$\lambda(r') \cdot \nu(-r') = \lambda(-r') \cdot \nu(r') = c. \tag{45}$$

Equivalently, wave quantities obey:

$$\lambda(r') = \frac{1}{\nu(-r')}, \quad \nu(r') = \frac{1}{\lambda(-r')}.$$

Proof. From $\lambda(r') \cdot \nu(r') = c$, taking log-inversion $r' \mapsto -r'$ implies:

$$\ln \lambda(-r') = -\ln \nu(r'), \Rightarrow \lambda(-r') = \frac{1}{\nu(r')}.$$

Then:

$$\lambda(r') \cdot \nu(-r') = \lambda(r') \cdot \frac{1}{\lambda(r')} = c.$$

□

13.4 Wave Inversion Illustration

As shown in Figure 3, the functions $\ln \lambda(r')$ and $\ln \nu(r')$ are symmetric reflections across $r' = 0$. The Schwarzschild radius thus acts as a physical scale-duality surface between high-frequency/short-wavelength and low-frequency/long-wavelength domains.

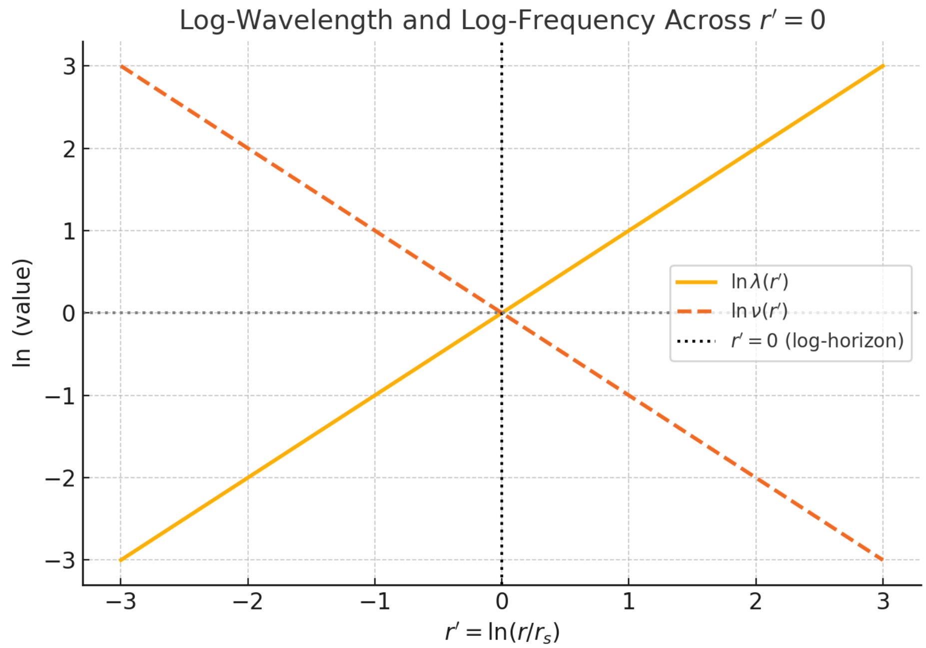

Figure 3: Logarithmic wavelength (solid line) and log-frequency (dashed line) plotted against log-radius $r' = \ln(r/r_s)$. At the duality surface $r' = 0$, corresponding to the Schwarzschild radius r_s, the wave observables invert: $\lambda(r') = 1/\nu(-r')$. This structure encodes T-duality-like behavior in log-spacetime.

13.5 Physical Interpretation and Duality Symmetry

The wave inversion identity implies:

- The Schwarzschild radius r_s is a scale-inversion horizon for physical observables.
- Interior and exterior wave modes are log-mirrored:

$$\lambda(r') \longleftrightarrow \frac{1}{\nu(r')}, \quad \nu(r') \longleftrightarrow \frac{1}{\lambda(r')}.$$

- Redshift and blueshift become geometric reflections in r'-space.

This reflects known dualities in high-energy physics:

- **T-duality:** $R \leftrightarrow 1/R$ in string theory,
- **UV/IR duality:** in holography and AdS/CFT,

- **Conformal inversion:** $x \mapsto 1/x$ in CFTs.

Remark 13.2. *This structure suggests that wave phenomena and quantum field modes may exhibit scale-coherent inversion dynamics — a novel feature of log-spacetime.*

14 Logarithmic Observer Duality and Physical Invariance

In logarithmic spacetime, where coordinates are defined by scale ratios, observers at reciprocal positions relative to a fixed scale (e.g., a Schwarzschild radius) experience mirrored but physically equivalent environments. This section formalizes and derives this duality.

14.1 Symmetric Log-Coordinates and Observer Pairing

Let r_s be a fixed reference length (e.g., Schwarzschild radius), and define:

$$r' := \ln\left(\frac{r}{r_s}\right), \quad \text{with inverse } r = r_s e^{r'}.$$

Now define two observers $\mathcal{O}_1$ and $\mathcal{O}_2$, located at:

$$r_1 = \frac{r_s}{\lambda}, \quad r_2 = \lambda r_s, \quad \lambda > 0,$$

so that their log-coordinates are:

$$r_1' = -\ln\lambda, \quad r_2' = \ln\lambda \quad \Rightarrow \quad r_1' = -r_2'.$$

Definition 14.1 (Observer Duality)**.** *Two observers are log-dual if their positions satisfy:*

$$r_1' = -r_2', \quad \textit{with } r_i' = \ln\left(\frac{r_i}{r_s}\right).$$

This reflects symmetry about the log-horizon $r' = 0$, the Schwarzschild reference.

14.2 Invariance of Measurable Quantities

We now show that several physical observables remain invariant or dual-symmetric under this log-inversion.

Theorem 14.2 (Gravitational Redshift Duality)**.** *Let a photon be emitted at position r_1 and received at r_2. In Schwarzschild geometry, the redshift is:*

$$z + 1 = \left(\frac{1 - r_s/r_1}{1 - r_s/r_2}\right)^{-1/2}.$$

If $r_1 = \frac{r_s}{\lambda}$ and $r_2 = \lambda r_s$, then:

$$z + 1 = \left(\frac{1-\lambda}{1-\lambda^{-1}}\right)^{-1/2} = \left(\frac{\lambda^{-1}-1}{1-\lambda^{-1}}\right)^{-1/2} = \lambda,$$

so that $\ln(z+1) = \ln\lambda = r'_2 = -r'_1$.

Corollary 14.3. *Redshift between log-dual observers is symmetric under scale inversion:*

$$z(r'_1 \to r'_2) = z(r'_2 \to r'_1)^{-1}.$$

14.3 Wave and Time Duality

Let a wave propagate radially. The wavelength and frequency transform as:

$$\lambda \cdot \nu = c \quad \Rightarrow \quad \ln\lambda + \ln\nu = \ln c.$$

If one observer measures log-wavelength $\lambda'(r')$, then its dual at $-r'$ measures:

$$\lambda'(-r') = -\nu'(r'), \quad \text{so } \lambda(r') = \frac{1}{\nu(-r')}.$$

This is consistent with previously defined wave inversion symmetry (cf. Section 13).

14.4 Field Theoretic Implication

A scalar field $\phi(r')$ propagating in log-space satisfies:

$$\Box_{\log}\phi = -\partial^2_{t'}\phi + \partial^2_{r'}\phi.$$

If $\phi(r')$ is a solution, then $\phi(-r')$ is also a solution in vacuum if the potential is symmetric:

$$V(\phi(r')) = V(\phi(-r')).$$

Theorem 14.4 (Log-Invariance of Scalar Fields)**.** *If $V(\phi)$ is even and $\phi(r')$ is a solution to the log-Klein–Gordon equation, then so is $\phi(-r')$.*

Proof. Direct substitution into the field equation and use of symmetry of V. □

14.5 Interpretation and Physical Consequences

- The Schwarzschild radius r_s acts as a physical inversion surface — a mirror between dual observers.
- All physical quantities measured in logarithmic coordinates behave symmetrically under $r' \mapsto -r'$.
- Observer duality replaces absolute scale with relative symmetry.

Remark 14.5. *This implies that regions inside and outside r_s are not fundamentally distinct — their physics is reciprocal under log inversion. This unifies black hole interiors and asymptotic regions under a common duality principle.*

15 Foundations of Logarithmic Spacetime

This section introduces the core geometric framework underlying the theory of logarithmic spacetime. Unlike standard formulations that assume linear coordinates and additive distances, we postulate that physical coordinates are inherently multiplicative, and that spacetime geometry is best represented on a logarithmic manifold.

15.1 Postulate: Multiplicative Spacetime Structure

We assume that physical distances and durations are best interpreted as ratios rather than differences. Thus, the natural coordinates of spacetime are logarithmic transforms of conventional ones:

Definition 15.1 (Logarithmic Coordinates). *Let $x > 0$ and $t > 0$ denote spatial and temporal coordinates relative to a reference scale. Define:*

$$x' := \ln\left(\frac{x}{x_s}\right), \quad t' := \ln\left(\frac{t}{t_s}\right),$$

where x_s and t_s are fixed reference units (e.g., Schwarzschild radius r_s and corresponding time $t_s = r_s/c$).

These coordinates linearize multiplicative transformations, such that spatial or temporal scaling corresponds to translation in log-space.

15.2 Logarithmic Metric Geometry

In these coordinates, we define a flat Lorentzian geometry:

Definition 15.2 (Logarithmic Metric). *Let $(t', x') \in \mathbb{R}^2$ be logarithmic time and space. Then the metric is:*

$$ds^2 = -dt'^2 + dx'^2,$$

describing a flat spacetime where scale changes behave analogously to displacements in special relativity.

This defines the underlying geometry of the manifold on which dynamics and curvature are formulated.

15.3 Scale Invariance and Geodesic Motion

Geodesics in log-spacetime preserve additive structure in (x', t'), which corresponds to multiplicative invariance in physical coordinates:

- Uniform motion in x' corresponds to exponential scaling in x,
- Geodesics trace logarithmic inflationary paths,

- Log-spacetime respects conformal scale transformations: $x' \mapsto x' + \lambda$.

Remark 15.3. *This formulation naturally encodes scale-invariant phenomena, such as cosmic inflation, renormalization group flow, and holographic duality, as straight-line dynamics on a flat manifold.*

15.4 Emergent Tiling from Factor Symmetries

Later sections will show how tiling structures emerge from the factor symmetries of integers. These tilings form geometric lattices in log-space, suggesting a discrete combinatorial structure on top of the flat background geometry.

Remark 15.4. *The log-square tiling discussed in Section 8.1 emerges not as an assumption, but as a consequence of number-theoretic structure superimposed on the log-metric.*

This foundational postulate — that spacetime is logarithmic in nature — underlies all subsequent constructions.

16 Physical Interpretation of Logarithmic Coordinates

To ground the theory of logarithmic spacetime in operational physics, we clarify the meaning and usage of log-coordinates such as $x' = \ln(x/x_0)$ and $t' = \ln(t/t_0)$, where x and t are physical intervals, and x_0, t_0 are observer-defined reference scales.

16.1 Scale-Relative Measurement

In this framework, physical quantities are expressed as ratios relative to a reference:

$$x' := \ln\left(\frac{x}{x_0}\right), \quad t' := \ln\left(\frac{t}{t_0}\right).$$

Then:

$$x' = 0 \Rightarrow x = x_0, \quad x' = 1 \Rightarrow x = e \cdot x_0, \quad x' = -1 \Rightarrow x = \frac{x_0}{e}.$$

This renders all physical measurements as multiplicative displacements from a defined scale, with x' counting the number of "e-foldings."

16.2 Observer-Defined Log Frames

Let an observer define a reference scale r_s (e.g., a Schwarzschild radius), and a reference time t_0. Then physical observables are defined as:

- $x' = \ln(r/r_s)$,
- $t' = \ln(t/t_0)$,
- $\nu' = \ln(\nu/\nu_0)$, with $\nu_0 = 1/t_0$,

- $E' = \ln(E/E_0)$.

All observables become additive under log-transformation, allowing unified scale-relative analysis.

Remark 16.1. *This formulation is aligned with conformal invariance and the logarithmic structure of thermodynamic and quantum field scaling.*

16.3 Logarithmic Intervals and Additive Translations

Intervals in log-space correspond to multiplicative ratios in physical space:

$$\Delta x' = \ln\left(\frac{x_2}{x_1}\right) \quad \Rightarrow \quad x_2 = e^{\Delta x'} x_1.$$

Exponential behavior becomes linear:

$$a(t) = a_0 e^{Ht} \quad \Rightarrow \quad \ln a = \ln a_0 + Ht.$$

Thus, log-space converts exponential growth into geodesic motion.

16.4 Duality and Inversion Symmetry

Logarithmic coordinates encode a duality:

$$x \leftrightarrow \frac{1}{x} \quad \Longleftrightarrow \quad x' \leftrightarrow -x'.$$

Definition 16.2 (Inversion Symmetric Point)**.** *The point $x' = 0$ corresponds to the self-dual scale $x = x_0$, which forms the origin of the log-manifold and serves as an observer's pivot scale.*

16.5 Physical Implications

- Local observers perceive ratios, not absolute magnitudes.
- Additive translations in x' correspond to multiplicative dilation.
- Hierarchies are logarithmically compressed.
- Dual observers at $r_1 = r_s/\lambda$, $r_2 = r_s\lambda$ are mirror images.

Remark 16.3. *Logarithmic spacetime is not a coordinate trick—it is a geometric framework for expressing physics in inherently scale-relative, symmetry-compatible form.*

17 The Role of Primes Beyond Number Theory

In logarithmic geometry, primes serve as physical markers of discrete symmetry-breaking, field coherence, and curvature minima.

17.1 Symmetry-Breaking in Log-Tiling Geometry

Let $\mathcal{F}(x) = \{f \in \mathbb{N} : 1 < f < x,\ f \mid x\}$. Then the log-map of each factor is:

$$P_f(x) := \ln\left(\frac{f}{\sqrt{x}}\right),$$

and satisfies:

$$P_{x/f}(x) = -P_f(x).$$

Definition 17.1 (Prime Symmetry-Breaking)**.** *A number x is prime if and only if $\mathcal{F}(x) = \emptyset$, indicating a break in the log-tiling symmetry.*

17.2 Prime Points as Field Resonators

Define a field resonance:

$$R_\alpha(x) := \sum_{f\in\mathcal{F}(x)} \cos\left(\alpha \ln\left(\frac{f}{\sqrt{x}}\right)\right).$$

Then for primes p, $R_\alpha(p) = 0$, and the field admits a pure mode:

$$\phi_p(x') = A\cos(\omega x' + \delta).$$

Remark 17.2. *Primes act as isolated resonance points for scale-invariant field modes.*

17.3 Curvature-Free Points in Discrete Geometry

Let $\ell_f := 2\,|\ln(f/\sqrt{x})|$. Then:

$$\mathcal{K}(x) := \sum_{(f,x/f)\in\mathcal{P}(x)} \frac{1}{\ell_f^2}.$$

Definition 17.3 (Curvature-Free Prime Node)**.** *If $x = p$ is prime, then $\mathcal{K}(p) = 0$. Prime points represent curvature-free positions in log-geometry.*

17.4 Primes as Quantum-Scale Eigenstates

Define a scale-based Hamiltonian:

$$H = -\frac{d^2}{dx'^2} + V(x'), \quad V(x') = -\lambda \sum_{p\in\mathbb{P}} \delta(x' - \ln p).$$

Definition 17.4 (Prime Eigenstate)**.** *A solution $\psi(x')$ of $H\psi = E\psi$ is a bound state if and only if it is localized at $x' = \ln p$, for $p \in \mathbb{P}$.*

17.5 Physical Summary

- Primes break tiling symmetry: they are log-isolated.
- Fields on primes are coherent and unperturbed.

- Curvature vanishes at primes.
- Quantum systems on log-lattices localize energy at primes.

Theorem 17.5. *In log-spacetime, primes are the only integers that simultaneously minimize:*

1. *Field interference,*
2. *Discrete curvature,*
3. *Spectral entropy.*

18 Extended Discrete Curvature in Logarithmic Geometry

18.1 Log-Tile Geometry and Edge Weights

Define log-tiles:

$$\mathcal{T}_x(f) := [\ln f, \ln(x/f)] \times [-s, s], \quad s := \left|\ln\left(\frac{f}{\sqrt{x}}\right)\right|.$$

Edge length:

$$\ell_f = 2s.$$

18.2 Regge-Like Discrete Curvature

Curvature at x is defined:

$$\mathcal{K}(x) := \sum_{(f, x/f) \in \mathcal{P}(x)} \frac{1}{\ell_f^2}.$$

Field Laplacian:

$$\Delta_{\log}\phi(x') := \sum_{f \in \mathcal{F}(x)} \frac{\phi(x'_f) - \phi(x')}{\ell_f^2}.$$

18.3 Logarithmic Action Functional

$$S[\phi] := \sum_{x \in \mathbb{N}} \left[\mathcal{K}(x)\phi^2(x') + \sum_{f \in \mathcal{F}(x)} \frac{(\phi(x'_f) - \phi(x'))^2}{\ell_f^2}\right].$$

18.4 Euler–Lagrange Equation

Stationarity yields:

$$\mathcal{K}(x)\phi(x') = \sum_{f \in \mathcal{F}(x)} \frac{\phi(x'_f) - \phi(x')}{\ell_f^2}.$$

Remark 18.1. *At primes $x = p$, this reduces to $\mathcal{K}(p) = 0$, hence $\phi(p')$ is unconstrained — a free eigenmode.*

Definition 18.2 (Discrete Log-Einstein Equation)**.**

$$\mathcal{K}(x) = T_{\log}(x), \quad T_{\log}(x) := \sum_{f \in \mathcal{F}(x)} \frac{(\phi(x'_f) - \phi(x'))^2}{\ell_f^2}.$$

19 Inflation as Inertial Motion: A Logarithmic Reinterpretation of Expansion

Conventional cosmology models the expansion of the universe via a time-dependent scale factor $a(t)$, governed by the Einstein–Friedmann equations. In this framework, inflationary epochs correspond to exponential growth in $a(t)$. We reinterpret this process geometrically using logarithmic spacetime, in which expansion corresponds not to metric stretching, but to uniform motion across a log-scale manifold.

19.1 Standard View: Exponential Scale Factor Evolution

In the FLRW metric,

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2,$$

the proper distance between comoving points grows as $a(t)$. During inflation:

$$a(t) = a_0 e^{Ht}, \quad H = \text{const.}$$

This implies:

$$\frac{\dot{a}(t)}{a(t)} = H, \quad \ddot{a}(t) > 0,$$

which defines an accelerated expansion regime in standard spacetime.

19.2 Logarithmic Coordinates and Linearization

Define log-scale coordinate:

$$x'(t) := \ln\left(\frac{a(t)}{a_0}\right) = \ln\left(e^{Ht}\right) = Ht.$$

The exponential scale factor becomes a linear trajectory:

$$x'(t) = Ht, \quad \frac{dx'}{dt} = H.$$

Proposition 19.1. *In logarithmic spacetime, exponential expansion becomes inertial motion along a geodesic in log-coordinates.*

Proof. Direct substitution of $a(t) = a_0 e^{Ht}$ into $x' = \ln(a/a_0)$ yields $x'(t) = Ht$, a straight line with constant velocity H. □

19.3 Geodesic Inflation and Observer Symmetry

In the flat log-metric:

$$ds^2 = -dt'^2 + dx'^2,$$

the evolution of the scale factor corresponds to a geodesic:

$$x'(t') = Ht', \quad \text{with } H = \text{constant in Planck units.}$$

This suggests:

- The universe evolves along a straight line in log-space.
- The expansion is not due to "stretching" but is intrinsic to scale-relative motion.
- Observer frames anchored to a_0 traverse $x'(t)$ without curvature.

19.4 Observable Consequences

- **CMB Uniformity**: Linear scale evolution may explain horizon uniformity without invoking rapid superluminal expansion.
- **Log-flat Power Spectrum**: Primordial perturbations are evenly distributed in x', predicting log-uniformity in scale.
- **Reheating Transitions**: Slope changes in $x'(t)$ may correspond to reheating epochs or phase transitions in the early universe.

19.5 Comparison Diagram

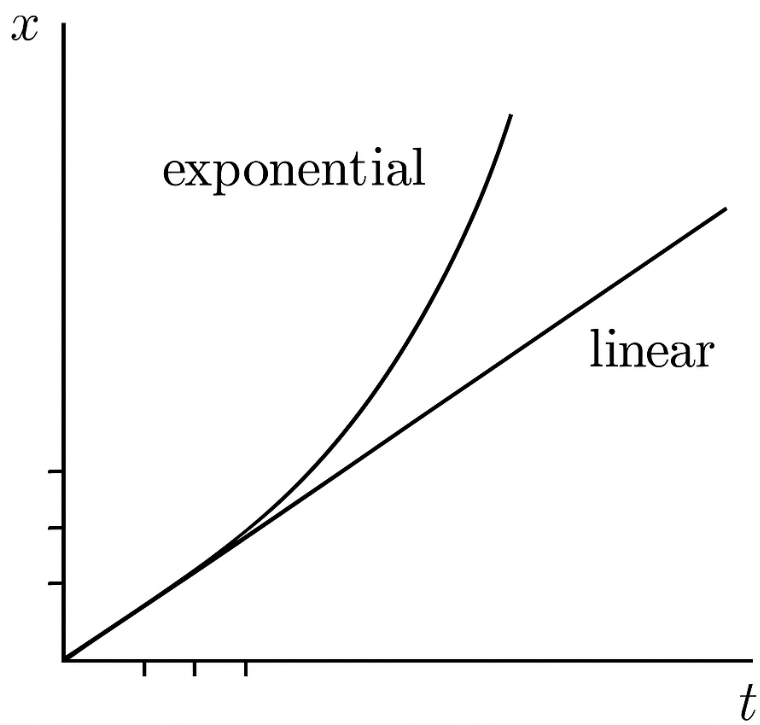

Figure 4: Exponential scale factor $a(t)$ (curve) vs. linear trajectory $x'(t)$ (line) in log-space.

Remark: Apparent Expansion in Logarithmic Spacetime

In standard coordinates, exponential scale growth $a(t) = a_0 e^{Ht}$ suggests that space itself expands — stretching distances between comoving observers. However, in logarithmic spacetime, this same evolution appears as uniform motion:

$$x'(t) = \ln\left(\frac{a(t)}{a_0}\right) = Ht,$$

which is linear and inertial in the log-coordinate x' as shown in Figure 4..

Interpretation: What appears as accelerated expansion in physical spacetime is simply geodesic translation in logarithmic scale. Space does not expand — it only seems to from a scale-agnostic perspective.

19.6 Relation to Previous Sections

This reformulation complements:

- Section 19, which modeled cosmic expansion using log-geodesics.
- Section 13, where field dynamics in log-time yielded inertial behavior.
- Section 9, predicting observable structure from log-uniform dynamics.

19.7 Conceptual Implications

- **No Absolute Expansion**: There is no "stretching fabric" — only a change of observational scale.
- **Time and Scale Equivalence**: Time evolution is geometrically identical to motion through scale.
- **Inertial Cosmology**: The inflationary phase is the default motion in flat log-spacetime.

19.8 Conclusion

This reinterpretation challenges the standard view of inflation as geometric acceleration. Instead, we frame it as inertial propagation through a log-coordinate system. The simplicity and symmetry of this model suggest that inflation may be a consequence of geometry rather than exotic field dynamics.

> *Inflation is not rapid expansion — it is motion through the fabric of scale.*

This reinterpretation of inflation as geodesic scale evolution provides a new lens for analyzing the early universe. In the following sections, we explore how this log-spacetime structure supports discrete curvature, arithmetic quantization, and field-theoretic behavior.

20 Reinterpreting Expansion in Logarithmic Spacetime

In standard cosmology, the expansion of space is modeled by the time evolution of the scale factor $a(t)$, governed by the Friedmann equations and interpreted as a stretching of spacetime itself. In contrast, logarithmic spacetime offers a reformulation in which exponential expansion becomes uniform motion in log-coordinates. This section explores the consequences of this reinterpretation.

20.1 Conventional Description of Expansion

In the Friedmann–Lemaître–Robertson–Walker (FLRW) metric:

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2,$$

cosmic expansion is captured by the scale factor $a(t)$. During inflation:

$$a(t) = a_0 e^{Ht},$$

for constant Hubble parameter H, implying exponential growth of proper distances between co-moving points.

20.2 Logarithmic Coordinates and Scale Motion

In logarithmic spacetime, define the log-scale coordinate:

$$x'(t) := \ln\left(\frac{a(t)}{a_0}\right) = \ln\left(e^{Ht}\right) = Ht.$$

This converts exponential expansion into linear motion:

$$x'(t) = Ht, \quad \frac{dx'}{dt} = H.$$

Thus, the evolution of $a(t)$ is inertial in log-space, and expansion corresponds to geodesic motion.

Proposition 20.1. *In logarithmic coordinates, exponential expansion $a(t) = a_0 e^{Ht}$ becomes uniform linear motion $x'(t) = Ht$, with constant velocity in log-space.*

20.3 Physical Interpretation

This reframing alters the ontology of expansion:

- **No stretching fabric**: Space itself is not "expanding" — rather, systems evolve uniformly through scale.
- **Inertial scale motion**: The universe evolves as if moving freely through a flat log-geometric manifold.
- **Geodesic inflation**: Inflation becomes a geodesic in log-spacetime, not an accelerated expansion.

Remark 20.2. *In log-spacetime, the expansion of the universe is not a physical stretching of space, but a scale-invariant shift in the observer's position on a multiplicative manifold.*

20.4 Implications for Cosmology

This reinterpretation leads to several conceptual and physical consequences:

- **Horizon problem**: Linear motion in log-space avoids the need for superluminal expansion to resolve causal horizons.
- **Entropy growth**: Logarithmic entropy becomes linear in time, simplifying thermodynamic modeling of the early universe.
- **Primordial modes**: Quantum fluctuations may appear as stationary waves in log-space, with resonance conditions governed by log-geometry.

20.5 Coordinate vs. Geometric Transformation

Unlike a mere change of variables, this reformulation modifies the geometric backdrop:

- In FLRW, ds^2 is defined on a manifold with dynamic scale factor $a(t)$.
- In log-spacetime, the metric is fixed and flat:

$$ds^2 = -dt'^2 + dx'^2,$$

  and all expansion is encoded in linear evolution over $x' = \ln a(t)$.

Theorem 20.3. *If expansion in standard cosmology is exponential, then in log-spacetime it becomes inertial motion on a flat manifold — implying that cosmological acceleration corresponds to uniform velocity in log-coordinates.*

Proof. Follows from direct substitution $x' := \ln a(t)$, yielding $x'(t) = Ht$ for exponential $a(t)$. □

20.6 Observable Predictions

- CMB anisotropies may appear log-uniform rather than scale-dependent.
- Large-scale structure correlations could reflect geodesic spacing in log-space.
- The reheating phase could correspond to a refraction in the log-geodesic trajectory.

20.7 Outlook

Recasting expansion as inertial motion in logarithmic spacetime provides a simplification of inflationary dynamics and invites alternative treatments of cosmological data. It suggests that the universe evolves along a preferred log-scale path, eliminating the need for dynamic geometry to model expansion.

> *What appears as exponential expansion in spacetime is straight-line motion in the geometry of scale.*

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21 Discussion and Future Work

- Formulate gauge and fermion fields in log-space.
- Develop path integral and quantization schemes.
- Explore connections to Möbius, totient, and zeta functions.
- Study log-tiling entropy and information networks.

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22 Conclusion

We have developed a geometric field theory in logarithmic spacetime, grounded in multiplicative symmetry and Schwarzschild referencing. Factor tilings, log-curvature, and dual observer frames define a framework rich with mathematical structure and physical insight — one that connects prime numbers to curvature, scale flow to geodesics, and entropy to log-symmetric tilings. This model invites both computational and observational inquiry into the very structure of scale itself.

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