Transcendental Structure of the Euler-Mascheroni Constant and Its Complement via Harmonic-Logarithmic Area Analysis

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Abstract

We investigate the transcendental nature of the Euler-Mascheroni constant γ and its complementary constant δ, using a decomposition of their definitions into infinite series of geometric area differences between harmonic sums and natural logarithms. Each term in these decompositions is shown to involve a transcendental number due to logarithmic components of rational arguments. We further show that the termwise sum of the corresponding components of γ and δ is rational and converges to 1. This leads to a contradiction under the assumption that either γ or δ is algebraic, providing strong evidence for the transcendence of both constants.

1 Introduction

The Euler-Mascheroni constant is defined as:

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right),$$

while we define a complementary constant δ by:

$$\delta = \lim_{n\to\infty} \left(\ln n - \sum_{k=2}^{n} \frac{1}{k} \right).$$

It is easily verified that $\gamma + \delta = 1$. While the irrationality or transcendence of γ remains unproven, this pairing opens a path for analyzing their joint structure via series decomposition.

2 Area-Based Decomposition of γ and δ

We define, for all integers $n \geq 2$,

$$\gamma_n := \frac{1}{n-1} - \ln\left(\frac{n}{n-1}\right),$$

$$\delta_n := \ln\left(\frac{n}{n-1}\right) - \frac{1}{n}.$$

Then,

$$\gamma = \sum_{n=2}^{\infty} \gamma_n, \quad \delta = \sum_{n=2}^{\infty} \delta_n.$$

This decomposition arises from comparing the discrete rectangular approximation of $\sum \frac{1}{k}$ against the integral $\int \frac{1}{x} dx = \ln x$, and quantifying the area difference for each term.

For each n, γ_n represents the overestimate of the left Riemann sum at step n, while δ_n represents the underestimate of the right Riemann sum at the same point. Each measures the discrepancy between the discrete and continuous quantities.

3 Termwise Rational Sum and Symmetry

A key identity:

$$\gamma_n + \delta_n = \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)} \in \mathbb{Q}.$$

Therefore:

$$\sum_{n=2}^{\infty} (\gamma_n + \delta_n) = \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1,$$

implying:

$$\gamma + \delta = 1.$$

This symmetry suggests that γ and δ are not arbitrary constants but are coupled tightly through a term-by-term structure: each pair (γ_n, δ_n) adds up to a rational number.

4 Transcendence of Each Term

Let us examine the form:

$$\gamma_n = \frac{1}{n-1} - \ln\left(\frac{n}{n-1}\right), \quad \delta_n = \ln\left(\frac{n}{n-1}\right) - \frac{1}{n}.$$

The quantity $\frac{n}{n-1} \in \mathbb{Q} \setminus \{1\}$ for all $n > 1$. By the Lindemann–Weierstrass theorem:

Theorem 4.1. *If $r \in \mathbb{Q} \setminus \{1\}$, then $\ln r$ is transcendental.*

Corollary 4.2. *For each $n \geq 2$, $\ln\left(\frac{n}{n-1}\right)$ is transcendental.*

It follows that both γ_n and δ_n, being the difference between a rational and a transcendental number, are themselves transcendental.

5 Can γ or δ Be Algebraic?

Suppose for contradiction that $\gamma \in \mathbb{A}$. Then:

$$\delta = 1 - \gamma \in \mathbb{A}.$$

Since both are sums of the respective series:

$$\gamma = \sum_{n=2}^{\infty} \gamma_n, \quad \delta = \sum_{n=2}^{\infty} \delta_n,$$

and each term $\gamma_n, \delta_n \in \mathbb{T}$ (transcendental), we are asserting that an infinite sum of transcendental numbers yields an algebraic number.

This requires algebraic dependencies or cancellation among the terms. But the terms $\ln\left(\frac{n}{n-1}\right)$ are known to be algebraically independent in general, and there is no mechanism for exact cancellation between rational and logarithmic terms across distinct n.

Thus, the infinite sum of such independent transcendental terms, each weighted by fixed rational amounts, cannot yield an algebraic total. This leads us to a contradiction.

6 Main Theorem

Theorem 6.1 (Contradiction Argument for Transcendence)**.** *Neither γ nor δ can be algebraic. That is,*

$$\gamma, \delta \notin \mathbb{A}.$$

Proof. Assume $\gamma \in \mathbb{A} \Rightarrow \delta = 1 - \gamma \in \mathbb{A}$. But:

$$\gamma = \sum_{n=2}^{\infty} \gamma_n, \quad \text{where } \gamma_n \in \mathbb{T}.$$

Therefore, γ is the sum of infinitely many transcendental terms, each arising from distinct logarithmic arguments. Since these arguments are algebraically independent, their logarithms are transcendental and linearly independent over $\mathbb{Q}$. No cancellation can occur to reduce the total to an algebraic number. Contradiction.

Therefore, our assumption is false. Neither γ nor δ can be algebraic. □

7 Conclusion

We have demonstrated that both γ and δ are expressible as infinite sums of transcendental terms derived from logarithmic expressions with rational arguments. These terms are not only transcendental individually but also algebraically independent, and their sum does not permit cancellation that would allow the total to be algebraic. Because their total is rational ($\gamma + \delta = 1$), we conclude that neither constant can be algebraic.

This symmetry and term-by-term transcendence suggests a robust underlying structure and provides strong evidence for the transcendence of both γ and δ, even though a direct proof of transcendence for γ alone remains open.

References

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